Metric-Independent Spacetime Volume-Forms and Dark Energy/Dark Matter Unification

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Abstract The method of non-Riemannian (metric-independent) spacetime volume-forms (alternative generally-covariant integration measure densities) is applied to construct a modified model of gravity coupled to a single scalar field providing an explicit unification of dark energy (as a dynamically generated cosmological constant) and dust fluid dark matter flowing along geodesics as an exact sum of two separate terms in the scalar field energy-momentum tensor. The fundamental reason for the dark species unification is the presence of a non-Riemannian volume-form in the scalar field action which both triggers the dynamical generation of the cosmological constant as well as gives rise to a hidden nonlinear Noether symmetry underlying the dust dark matter fluid nature. Upon adding appropriate perturbation breaking the hidden "dust" Noether symmetry we preserve the geodesic flow property of the dark matter while we suggest a way to get growing dark energy in the present universe' epoch free of evolution pathologies. Also, an intrinsic relation between the above modified gravity + single scalar field model and a special quadratic purely kinetic "k-essence" model is established as a weak-versus-strong-coupling duality.

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1 Introduction

According to the standard cosmological model (ΛCDM model [1]-[3]) the energy density of the late time Universe is dominated by two "dark" components - around 70 % made out of "dark energy" [4]-[6] and around 25 % made out of "dark matter" [7]-[9]. Since more than a decade a principal challenge in modern cosmology is to understand theoretically from first principles the nature of both "dark" species of the universe's substance as a manifestation of the dynamics of a single entity of matter. Among the multitude of approaches to this seminal problem proposed so far are the (generalized) "Chaplygin gas" models [10]-[13], the "purely kinetic k-essence" models [18]-[21] based on the class of kinetic "quintessence" models [14]-[17], and more recently – the so called "mimetic" dark matter model [22, 23] and its extensions [24, 25], as well as constant-pressure-ansatz models [26].

Here we will describe a new approach achieving unified description of dark energy and dark matter based on a class of generalized models of gravity interacting with a single scalar field employing the method of non-Riemannian volume-forms on the pertinent spacetime manifold [27]-[31] (for further developments, see Refs.[32, 33]). Non-Riemannian spacetime volume-forms or, equivalently, alternative generally covariant integration measure densities are defined in terms of auxiliary maximal-rank antisymmetric tensor gauge fields ("measure gauge fields") unlike the standard Riemannian integration measure density given given in terms of the square root of the determinant of the spacetime metric. These non-Riemannian-measure-modified gravity-matter models are also called "two-measure gravity theories".

Let us particularly stress that the method of non-Riemannian spacetime volume-forms is a very powerful one having profound impact in any (field theory) models with general coordinate reparametrization invariance, such as general relativity and its extensions [27]-[31], [32]-[36], [37]-[39]; strings and (higher-dimensional) membranes [40, 41]; and supergravity [42, 43]. Among its main features we should mention:

- Dynamical generation of cosmological constant as arbitrary integration constant in the solution of the equations of motion for the auxiliary "measure" gauge fields (see also Eq.(6) below).
- Using the canonical Hamiltonian formalism for Dirac-constrained systems we find that the auxiliary "measure" gauge fields are in fact almost pure gauge degrees of freedom except for the above mentioned arbitrary integration constants which are identified with the conserved Dirac-constrained canonical momenta conjugated to the "magnetic" components of the "measure" gauge fields [38, 39].
- Applying the non-Riemannian volume-form formalism to minimal $N = 1$ supergravity the appearance of a dynamically generated cosmological constant triggers spontaneous supersymmetry breaking and mass generation for the gravitino (supersymmetric Brout-Englert-Higgs effect) [42, 43]. Applying the same formalism to anti-de Sitter supergravity allows to produce simultaneously a very large physical gravitino mass and a very small *positive* observable cosmological

constant [42, 43] in accordance with modern cosmological scenarios for slowly expanding universe of the present epoch [4, 5, 6].

- Employing two independent non-Riemannian volume-forms produces effective scalar potential with two infinitely large flat regions [37, 38] (one for large negative and another one for large positive values of the scalar field φ) with vastly different scales appropriate for a unified description of both the early and late universe' evolution. A remarkable feature is the existence of a stable initial phase of *non-singular* universe creation preceding the inflationary phase – stable "emergent universe" without "Big-Bang" [37].

In Section 2 below we briefly discuss a non-standard model of gravity interacting with a single scalar field which couples symmetrically to a standard Riemannian as well as to another non-Riemannian volume form (spacetime integration measure density). We show that the auxiliary "measure" gauge field dynamics produces an arbitrary integration constant identified as a dynamically generated cosmological constant giving rise to a the dark energy term in the pertinent energy-momentum tensor. Simultaneously, a hidden strongly nonlinear Noether symmetry of the scalar Lagrangian action is revealed leading to a "dust" fluid representation of the second term in the energy-momentum tensor, which accordingly is identified as a "dust" dark matter flowing along geodesics. Thus, both "dark" species are explicitly unified as an exact sum of two separate contributions to the energy-momentum tensor.

In Section 3 some implications for cosmology are briefly considered. Specifically, we briefly study an appropriate perturbation of our modified-measure gravity + scalar-field model which breaks the above crucial hidden Noether symmetry and introduces exchange between the dark energy and dark matter components, while preserving the geodesic flow property of the dark matter fluid. Further, we suggest how to obtain a growing dark energy in the present day universe' epoch without invoking any pathologies of "cosmic doomsday" or future singularities kind [44]-[46].

In Sections 4 below we couple the above modified-measure scalar-field model to a quadratic $f(R)$-gravity. We derive the pertinent "Einstein"-frame effective theory which turns out be a very special quadratic purely kinetic "k-essence" gravity-matter model. The main result here is establishing duality (in the standard sense of weak versus strong coupling) between the latter and the original quadratic $f(R)$-gravity plus modified-measure scalar-field model, whose matter part delivers an exact unified description of dynamical dark energy and dust fluid dark matter.

Section 5 contains our concluding remarks.

For further details, in particular, canonical Hamiltonian treatment and Wheeler-DeWitt quantization of the above unified model of dark energy and dark matter, see Refs.[36, 47].

2 Gravity-Matter Theory With a Non-Riemannian Volume-Form in the Scalar Field Action – Hidden Noether Symmetry and Unification of Dark Energy and Dark Matter

Let us consider the following simple particular case of a non-conventional gravity-scalar-field action – a member of the general class of the "two-measure" gravity-matter theories [28]-[31] (for simplicity we use units with the Newton constant $G_N = 1/16\pi$):

$$S = \int d^4x\sqrt{-g}R + \int d^4x\big(\sqrt{-g}+\Phi(B)\big)L(\varphi,X) \; . \tag{1}$$

Here R denotes the standard Riemannian scalar curvature for the pertinent Riemannian metric $g_{\mu\nu}$. The second term in (1) – the scalar field action is constructed in terms of two mutually independent spacetime volume-forms (integration measure densities):

(a) $\sqrt{-g} \equiv \sqrt{-\det\|g_{\mu\nu}\|}$ is the standard Riemannian integration measure density;

(b) $\Phi(B)$ denotes an alternative non-Riemannian generally covariant integration measure density independent of $g_{\mu\nu}$ and defining an alternative non-Riemannian volume-form:

$$\Phi(B) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu B_{\nu\kappa\lambda} \; , \tag{2}$$

where $B_{\mu\nu\lambda}$ is an auxiliary maximal rank antisymmetric tensor gauge field independent of the Riemannian metric, also called "measure gauge field".

$L(\varphi,X)$ is general-coordinate invariant Lagrangian of a single scalar field $\varphi(x)$, the simplest example being:

$$L(\varphi,X) = X - V(\varphi) \quad , \quad X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \; , \tag{3}$$

As it will become clear below, the final result about the unification of dark energy and dark matter resulting from an underlying hidden Noether symmetry (see (9) below) of the scalar field action (second term in (1)) does *not* depend on the detailed form of $L(\varphi,X)$ which could be of an arbitrary generic "k-essence" form [14]-[17]:

$$L(\varphi,X) = \sum_{n=1}^{N} A_n(\varphi)X^n - V(\varphi) \; , \tag{4}$$

i.e., a nonlinear (in general) function of the scalar kinetic term X.

Due to general-coordinate invariance we have covariant conservation of the scalar field energy-momentum tensor:

$$T_{\mu\nu} = g_{\mu\nu}L(\varphi,X) + \Big(1+\frac{\Phi(B)}{\sqrt{-g}}\Big)\frac{\partial L}{\partial X}\partial_\mu\varphi\,\partial_\nu\varphi \quad , \quad \nabla^\nu T_{\mu\nu} = 0 \qquad . \tag{5}$$

Equivalently, energy-momentum conservation (5) follows from the second-order equation of motion w.r.t. φ. The latter, however, becomes redundant because the modified-measure scalar field action (second term in (1)) exhibits a crucial new property – it yields a *dynamical constraint* on $L(\varphi,X)$ as a result of the equations of motion w.r.t. "measure" gauge field $B_{\mu\nu\lambda}$:

$$\partial_\mu L(\varphi,X) = 0 \quad \longrightarrow \quad L(\varphi,X) = -2M = \text{const} \,, \tag{6}$$

in particular, for (3):

$$X - V(\varphi) = -2M \;\; \longrightarrow \;\; X = V(\varphi) - 2M \,, \tag{7}$$

where M is arbitrary integration constant. The factor 2 in front of M is for later convenience, moreover, we will take $M > 0$ in view of its interpretation as a dynamically generated cosmological constant[1]. Indeed, taking into account (6), the expression (5) becomes:

$$T_{\mu\nu} = -2M g_{\mu\nu} + \Big(1 + \frac{\Phi(B)}{\sqrt{-g}}\Big) \frac{\partial L}{\partial X} \partial_\mu \varphi \partial_\nu \varphi \,. \tag{8}$$

As already shown in Ref.[36] the scalar field action in (1) possesses a hidden strongly nonlinear Noether symmetry, namely (1) is invariant (up to a total derivative) under the following nonlinear symmetry transformations:

$$\delta_\varepsilon \varphi = \varepsilon \sqrt{X} \quad , \quad \delta_\varepsilon g_{\mu\nu} = 0 \quad , \quad \delta_\varepsilon \mathscr{B}^\mu = -\varepsilon \frac{1}{2\sqrt{X}} g^{\mu\nu} \partial_\nu \varphi \big(\Phi(B) + \sqrt{-g}\big) \,, \tag{9}$$

where $\mathscr{B}^\mu \equiv \frac{1}{3!} \varepsilon^{\mu\nu\kappa\lambda} B_{\nu\kappa\lambda}$. Under (9) the action (1) transforms as
$\delta_\varepsilon S = \int d^4x \partial_\mu \big(L(\varphi,X) \delta_\varepsilon \mathscr{B}^\mu\big)$. Then, the standard Noether procedure yields the conserved current:

$$\nabla_\mu J^\mu = 0 \quad , \quad J^\mu \equiv \Big(1 + \frac{\Phi(B)}{\sqrt{-g}}\Big) \sqrt{2X} g^{\mu\nu} \partial_\nu \varphi \frac{\partial L}{\partial X} \,. \tag{10}$$

$T_{\mu\nu}$ (8) and J^μ (10) can be cast into a relativistic hydrodynamical form:

$$T_{\mu\nu} = -2M g_{\mu\nu} + \rho_0 u_\mu u_\nu \quad , \quad J^\mu = \rho_0 u^\mu \,, \tag{11}$$

where:

$$\rho_0 \equiv \Big(1 + \frac{\Phi(B)}{\sqrt{-g}}\Big) 2X \frac{\partial L}{\partial X} \;\; , \;\; u_\mu \equiv \frac{\partial_\mu \varphi}{\sqrt{2X}} \;\; , \;\; u^\mu u_\mu = -1 \,. \tag{12}$$

For the pressure p and energy density ρ we have accordingly (with ρ_0 as in (12)):

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[1] The physical meaning of the "measure" gauge field $B_{\mu\nu\lambda}$ (2) as well as the meaning of the integration constant M are most straightforwardly seen within the canonical Hamiltonian treatment of (1) [36]. For more details about the canonical Hamiltonian treatment of general gravity-matter theories with (several independent) non-Riemannian volume-forms we refer to [38, 39].

$$p = -2M = \text{const} \quad , \quad \rho = \rho_0 - p = \Bigl(1 + \frac{\Phi(B)}{\sqrt{-g}}\Bigr) 2X \frac{\partial L}{\partial X} + 2M \; , \tag{13}$$

where the integration constant M appears as *dynamically generated cosmological constant*.

Thus, $T_{\mu\nu}$ (11) represents an exact sum of two contributions of the two dark species with $p = p_{\rm DE} + p_{\rm DM}$ and $\rho = \rho_{\rm DE} + \rho_{\rm DM}$:

$$p_{\rm DE} = -2M \;,\; \rho_{\rm DE} = 2M \quad ; \quad p_{\rm DM} = 0 \;,\; \rho_{\rm DM} = \rho_0 \; , \tag{14}$$

i.e., the dark matter component is a dust fluid ($p_{\rm DM} = 0$).

Covariant conservation of $T_{\mu\nu}$ (11) immediately implies *both* (i) the covariant conservation of $J^\mu = \rho_0 u^\mu$ (10) describing dust dark matter "particle number" conservation, and (ii) the geodesic flow equation of the dust dark matter fluid:

$$\nabla_\mu \bigl(\rho_0 u^\mu\bigr) = 0 \quad , \quad u_\nu \nabla^\nu u_\mu = 0 \; . \tag{15}$$

3 Some Cosmological Implications

Let us now consider a perturbation of the initial modified-measure gravity + scalar-field action (1) by some additional scalar field Lagrangian $\widehat{L}(\varphi,X)$ independent of the initial scalar Lagrangian $L(\varphi,X)$:

$$\widehat{S} = \int d^4x \sqrt{-g} R + \int d^4x \bigl(\sqrt{-g} + \Phi(B)\bigr) L(\varphi,X) + \int d^4x \sqrt{-g} \widehat{L}(\varphi,X) \; . \tag{16}$$

An important property of the perturbed action (16) is that once again the scalar field φ-dynamics is given by the unperturbed dynamical constraint Eq.(6) of the initial scalar Lagrangian $L(\varphi,X)$, which is completely independent of the perturbing scalar Lagrangian $\widehat{L}(\varphi,X)$.

Henceforth, for simplicity we will take the scalar Lagrangians in the canonical form $L(\varphi,X) = X - V(\varphi)$, $\widehat{L}(\varphi,X) = X - U(\varphi)$, where $U(\varphi)$ is independent of $V(\varphi)$.

The associated scalar field energy-momentum tensor now reads (cf. Eqs.(11)-(13)):

$$\widehat{T}_{\mu\nu} = \widehat{\rho}_0 u_\mu u_\nu + g_{\mu\nu}\bigl(-4M + V - U\bigr) \quad , \quad \widehat{\rho}_0 \equiv 2(V - 2M)\Bigl(1 + \frac{\Phi(B)}{\sqrt{-g}}\Bigr) \; , \tag{17}$$

or, equivalently:

$$\widehat{T}_{\mu\nu} = \bigl(\widehat{\rho} + \widehat{p}\bigr) u_\mu u_\nu + \widehat{p}\, g_{\mu\nu} \quad , \quad \widehat{p} = -4M + V - U \; , \tag{18}$$

$$\widehat{\rho} = \widehat{\rho}_0 - \widehat{p} = 2(V - 2M)\Bigl(1 + \frac{\Phi(B)}{\sqrt{-g}}\Bigr) + 4M + U - V \; , \tag{19}$$

where (7) is used.

The perturbed energy-momentum (17) conservation $\nabla^{\mu}\widehat{T}_{\mu\nu}=0$ now implies:

- The perturbed action (16) does not any more possess the hidden symmetry (9) and, therefore, the conservation of the dust particle number current $J^{\mu}=\rho_0 u^{\mu}$ (11) is now replaced by:

$$\nabla^{\mu}\left(\widehat{\rho}_0 u_{\mu}\right)+\sqrt{2(V-2M)}\Big(\frac{\partial V}{\partial\varphi}-\frac{\partial U}{\partial\varphi}\Big)=0\,. \tag{20}$$

- *Once again* we obtain the geodesic flow equation for the dark matter "fluid" (second Eq. (15)). Let us stress that this is due to the fact that the perturbed pressure $\widehat{p}$ (second relation in (18)), because of the dynamical constraint (7) triggered by the non-Riemannian volume-form in (16), is a function of φ only but not of X.

Thus, we conclude that the geodesic flow dynamics of the cosmological fluid described by the action (16) persist irrespective of the presence of the perturbation (last term in (16)) as well as of the specific form of the latter.

In the cosmological context, when taking the spacetime metric in the standard Friedmann-Lemaitre-Robertson-Walker (FLRW) form, the scalar field is assumed to be time-dependent only: $\varphi=\varphi(t)$. Thus, in this case the dynamical constraint Eq.(7) and its solution assume the form:

$$\dot{\varphi}^2=2\big(V(\varphi)-2M\big)\quad\longrightarrow\quad\int_{\varphi(0)}^{\varphi(t)}\frac{d\varphi}{\sqrt{2\big(V(\varphi)-2M\big)}}=\pm t\,. \tag{21}$$

Choosing the $+$ sign in (21) corresponds to $\varphi(t)$ monotonically growing with t irrespective of the detailed form of the potential $V(\varphi)$. The only condition due to consistency of the dynamical constraint (first Eq.(21)) is $V(\varphi)>2M$ for the whole interval of classically accessible values of φ. Also, note the "strange" looking second-order (in time derivatives) form of the first Eq.(21): $\ddot{\varphi}-\partial V/\partial\varphi=0$, where we specifically stress on the *opposite* sign in the force term. Thus, it is fully consistent for $\varphi(t)$ to "climb" a growing w.r.t. φ scalar potential.

As already stressed above, the dynamics of the $\varphi(t)$ does not depend at all on the presence of the perturbing scalar potential $U(\varphi)$. Therefore, if we choose the perturbation $U(\varphi)$ in (16) such that the potential difference $U(\varphi)-V(\varphi)$ is a growing function at large φ (*e.g.*, $U(\varphi)-V(\varphi)\sim e^{\alpha\varphi}$, α small positive) then, when $\varphi(t)$ evolves through (21) to large positive values, it (slowly) "climbs" $U(\varphi)-V(\varphi)$ and according to the expression $\widehat{\rho}_{DE}=4M+U(\varphi)-V(\varphi)=-\widehat{p}$ for the dark energy density (cf. (17)-(18)), the latter will (slowly) grow up! Let us emphasize that in this way we obtain growing dark energy of the "late" universe without any pathologies in the universe' evolution like "cosmic doomsday" or future singularities [44]-[46].

4 Duality to Purely Kinetic "K-Essence"

Let us now consider a different perturbation of the modified-measure gravity + scalar-field action (1) by replacing the standard Einstein-Hilbert gravity action (the first term in (1)) with a $f(R) = R - \alpha R^2$ extended gravity action in the first-order Palatini formalism:

$$S^{(\alpha)} = \int d^4x \sqrt{-g}\big(R(g,\Gamma) - \alpha R^2(g,\Gamma)\big) + \int d^4x \big(\sqrt{-g} + \Phi(B)\big) L(\varphi, X) \;, \quad (22)$$

where $R(g,\Gamma) = g^{\mu\nu} R_{\mu\nu}(\Gamma)$, *i.e.*, with *a priori* independent metric $g_{\mu\nu}$ and affine connection $\Gamma^{\mu}_{\nu\lambda}$.

Since the scalar field action – the second term in (22) – remains the same as in the original action (1), and the hidden nonlinear Noether symmetry (9) does not affect the metric, all results in Section 2 remain valid. Namely, the Noether symmetry (9) produces "dust" fluid particle number conserved current (first Eq.(15)) and interpretation of φ as describing simultaneously dark energy (because of the dynamical scalar Lagrangian constraint (6)) and dust dark matter with geodesic dust fluid flow (second Eq.(15)) remains intact.

However, the gravitational equations of motion derived from (22) are not of the standard Einstein form:

$$R_{\mu\nu}(\Gamma) = \frac{1}{2f'_R}\left[T_{\mu\nu} + f(R) g_{\mu\nu}\right] \;, \quad (23)$$

where $f(R) = R(g,\Gamma) - \alpha R^2(g,\Gamma)$, $f'_R = 1 - 2\alpha R(g,\Gamma)$ and $T_{\mu\nu}$ is the same as in (8).

The equations of motion w.r.t. independent $\Gamma^{\mu}_{\nu\lambda}$ resulting from (22) yield (for an analogous derivation, see [28]) the following solution for $\Gamma^{\mu}_{\nu\lambda}$ as a Levi-Civita connection:

$$\Gamma^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda}(\bar{g}) = \frac{1}{2}\bar{g}^{\mu\kappa}\left(\partial_\nu \bar{g}_{\lambda\kappa} + \partial_\lambda \bar{g}_{\nu\kappa} - \partial_\kappa \bar{g}_{\nu\lambda}\right) \;, \quad (24)$$

w.r.t. to the Weyl-rescaled metric $\bar{g}_{\mu\nu}$:

$$\bar{g}_{\mu\nu} = f'_R g_{\mu\nu} \;, \quad (25)$$

so that $\bar{g}_{\mu\nu}$ is called (physical) "Einstein-frame" metric. In passing over to the "Einstein-frame" it is also useful to perform the following φ-field redefinition:

$$\varphi \to \widetilde{\varphi} = \int \frac{d\varphi}{\sqrt{\big(V(\varphi) - 2M\big)}} \quad , \quad X \to \widetilde{X} = -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu \widetilde{\varphi} \partial_\nu \widetilde{\varphi} = \frac{1}{f'_R} \;, \quad (26)$$

where the last relation follows from the Lagrangian dynamical constraint (7) together with (25).

Derivation of the explicit expressions for the Einstein-frame gravitational equations, *i.e.*, equations w.r.t. Einstein-frame metric (25) and the Einstein-frame scalar field (first Eq.(26)), yields the latter in the standard form of Einstein gravity equations:

$$\overline{R}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{R} = \frac{1}{2}\overline{T}_{\mu\nu} \; . \tag{27}$$

Here the following notations are used:

(i) $\overline{R}_{\mu\nu}$ and $\overline{R}$ are the standard Ricci tensor and scalar curvature of the Einstein-frame metric (25).

(ii) The Einstein-frame energy-momentum tensor:

$$\overline{T}_{\mu\nu} = \overline{g}_{\mu\nu}\overline{L}_{\rm eff} - 2\frac{\partial \overline{L}_{\rm eff}}{\partial \overline{g}^{\mu\nu}} \tag{28}$$

is given in terms of the following effective $\widetilde{\varphi}$-scalar field Lagrangian of a specific quadratic purely kinetic "k-essence" form:

$$\overline{L}_{\rm eff}(\widetilde{X}) = \Big(\frac{1}{4\alpha} - 2M\Big)\widetilde{X}^2 - \frac{1}{2\alpha}\widetilde{X} + \frac{1}{4\alpha} \; . \tag{29}$$

Thus, the Einstein-frame gravity+scalar-field action reads:

$$S_{\rm k-ess} = \int d^4\sqrt{-\overline{g}}\Big[\overline{R} + \Big(\frac{1}{4\alpha} - 2M\Big)\widetilde{X}^2 - \frac{1}{2\alpha}\widetilde{X} + \frac{1}{4\alpha}\Big] \; . \tag{30}$$

The Einstein-frame effective energy-momentum-tensor (28) in the perfect fluid representation reads (taking into account the explicit form of $\overline{L}_{\rm eff}$ (29)):

$$\bar{T}_{\mu\nu} = \overline{g}_{\mu\nu}\widetilde{p} + \widetilde{u}_\mu\widetilde{u}_\nu\big(\widetilde{\rho} + \widetilde{p}\big) \quad , \quad \widetilde{u}_\mu \equiv \frac{\partial_\mu\widetilde{\varphi}}{\sqrt{2\widetilde{X}}} \; , \; \overline{g}^{\mu\nu}\widetilde{u}_\mu\widetilde{u}_\nu = -1 \; , \tag{31}$$

$$\widetilde{p} = \Big(\frac{1}{4\alpha} - 2M\Big)\widetilde{X}^2 - \frac{1}{2\alpha}\widetilde{X} + \frac{1}{4\alpha} \; , \; \widetilde{\rho} = 3\Big(\frac{1}{4\alpha} - 2M\Big)\widetilde{X}^2 - \frac{1}{2\alpha}\widetilde{X} - \frac{1}{4\alpha} \; . \tag{32}$$

Let us stress that the quadratic purely kinetic "k-essence" scalar Lagrangian (29) is indeed a very special one:

- The three coupling constants in (29) depend only on two independent parameters (α, M), the second one being a dynamically generated integration constant in the original theory (22).
- The quadratic gravity term $-\alpha R^2$ in (22) is just a small perturbation w.r.t. the initial action (1) when $\alpha \to 0$, whereas the coupling constants in the Einstein-frame effective action (30) diverge as $1/\alpha$, *i.e.*, weak coupling in (22) is equivalent to a strong coupling in (30).
- Due to the apparent Noether symmetry of (29) under constant shift of $\widetilde{\varphi}$ ($\widetilde{\varphi} \to \widetilde{\varphi} + \text{const}$) the corresponding Noether conservation law is identical to the $\widetilde{\varphi}$-equations of motion:

$$\overline{\nabla}_\mu\Big(\overline{g}^{\mu\nu}\partial_\nu\widetilde{\varphi}\frac{\partial\widetilde{L}_{\rm eff}}{\partial\widetilde{X}}\Big) = 0 \; , \tag{33}$$

where $\overline{\nabla}_\mu$ is covariant derivative w.r.t. the Levi-Civita connection (24) in the $\bar{g}_{\mu\nu}$-(Einstein) frame. Eq.(33) is the Einstein-frame counterpart of the "dust" Noether conservation law (10) in the original theory (1) or (22).

Thus, we have found an explicit duality in the usual sense of "weak versus strong coupling" between the original non-standard gravity+scalar-field model providing exact unified description of dynamical dark energy and dust fluid dark matter in the matter sector, on one hand, and a special quadratic purely kinetic "k-essence" gravity-matter model, on the other hand. The latter dual theory arises as the "Einstein-frame" effective theory of its original counterpart.

To make explicit the existence of smooth strong coupling limit $\alpha \to 0$ *on-shell* in the dual "k-essence" energy density $\widetilde{\rho}$ and "k-essence" pressure $\widetilde{p}$ (32) in spite of the divergence of the corresponding constant coefficients, let us consider a reduction of the dual quadratic purely kinetic "k-essence" gravity + scalar-field model (30) for the Friedmann-Lemaitre-Robertson-Walker (FLRW) class of metrics:

$$ds^2 = -N^2(t)dt^2 + a^2(t)\Big[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big] . \tag{34}$$

The FLRW reduction of the $\phi \equiv \widetilde{\varphi}$-equation of motion (33) (using henceforth the gauge $N = 1$) reads:

$$\frac{dp_\phi}{dt} = 0 \;\; \longrightarrow \;\; p_\phi = a^3\Big[-\frac{1}{2\alpha}\,\dot{\phi} + \big(\frac{1}{4\alpha} - 2M\big)\,\dot{\phi}^3\Big] , \tag{35}$$

where p_ϕ is the constant conserved canonically conjugated momentum of $\phi \equiv \widetilde{\varphi}$. Thus, the velocity $\dot{\phi} = \dot{\phi}\,(p_\phi/a^3)$ is a function of the Friedmann scale factor $a(t)$ through the ratio p_ϕ/a^3 and solves the cubic algebraic equation (35) for any α. For small α we get:

$$\dot{\phi}\;(p_\phi/a^3) \simeq \sqrt{2} + \alpha\Big(4\sqrt{2}M + \frac{p_\phi}{a^3}\Big) + \mathrm{O}(\alpha^2) . \tag{36}$$

Then, inserting (36) into the FLRW-reduced $\widetilde{X} = \frac{1}{2}\,\dot{\phi}^2$ and substituting it into the expressions (32) we obtain for the small-α asymptotics of the "k-essence" energy density and "k-essence" pressure:

$$\widetilde{\rho} = 2M + \sqrt{2}\frac{p_\phi}{a^3} + \alpha\Big[16M^2 + 4\sqrt{2}M\frac{p_\phi}{a^3} + \frac{1}{2}\big(\frac{p_\phi}{a^3}\big)^2\Big] + \mathrm{O}(\alpha^2) , \tag{37}$$

$$\widetilde{p} = -2M - \alpha\Big[16M^2 - \frac{1}{2}\big(\frac{p_\phi}{a^3}\big)^2\Big] + \mathrm{O}(\alpha^2) . \tag{38}$$

The limiting values $\widetilde{\rho} = 2M + \sqrt{2}\frac{p_\phi}{a^3}$ and $\widetilde{p} = -2M$ precisely coincide with the corresponding values of ρ and p (13) in the FLRW reduced original theory (1) [36].

5 Conclusions

In the present note we have demonstrated the power of the method of non-Riemannian spacetime volume-forms (alternative generally-covariant integration measure densities) by applying it to construct a modified model of gravity coupled to a single scalar field which delivers a unification of dark energy (as a dynamically generated cosmological constant) and dust fluid dark matter flowing along geodesics (due to a hidden nonlinear Noether symmetry). Both "dark" species appear as an exact sum of two separate contributions in the energy-momentum tensor of the single scalar field. Upon perturbation of the scalar field action, which breaks the hidden "dust" Noether symmetry but preserves the geodesic flow property, we show how to obtain a growing dark energy in the late Universe without evolution pathologies. Furthermore, we have established a duality (in the standard sense of weak versus strong coupling) of the above model unifying dark energy and dark matter, on one hand, and a specific quadratic purely kinetic "k-essence" model. This duality elucidates the ability of purely kinetic "k-essence" theories to describe approximately the unification of dark energy and dark matter and explains how the "k-essence" description becomes exact in the strong coupling limit on the "k-essence" side.

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